

# Black Hole Probes of Automorphic Space

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## Abstract

Over the past few years the arithmetic Langlands program has proven useful in addressing physical problems. In this paper it is shown how Langlands' reciprocity conjecture for automorphic forms, in combination with a representation theoretic notion of motives, suggests a framework in which the entropy of automorphic black holes can be viewed as a probe of space-time that is sensitive to the geometry of the extra dimensions predicted by string theory. If it were possible to produce black holes with automorphic entropy in the laboratory their evaporation would provide us with information about the precise shape of the compact geometry.

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# 1 Introduction

Black holes have played an essential role in developing our understanding of fundamental aspects of gravity. In particular their role as thermodynamical objects has been one of the central issues ever since the paradigmatic insights of Bekenstein and Hawking. More recently, the structure of black holes with higher supersymmetry has been illuminated in fundamentally new ways by the application of tools from the theory of modular forms of higher rank. The emergence of such automorphic forms makes it possible to link the theory of supersymmetric black holes to the arithmetic Langlands program [1], thereby making it possible to introduce ideas from the theory of automorphic representations as new tools that can be used to elucidate the implications of the entropy results obtained so far.

One of the key questions raised by the Langlands program is whether general algebraic automorphic forms are geometric in the sense that there exist geometric structures, called motives, whose properties encode exactly the same information as the automorphic form. The roots of this problem reach back into the late 19th century, but only its modern incarnation as the framework of automorphic motives is far-reaching enough to have found an application in string theory, in particular the problem of an emergent spacetime [2]. The emergence of automorphic black hole entropy functions makes it therefore natural to ask whether they encode motivic information about the compact part of spacetime. This leads to a new point of view of (automorphic) black holes as physical probes that are sensitive to the nature of the extra dimensions. The automorphic nature of the entropy implies that if one were to experiment with such black holes in the laboratory, a finite number of measurements would suffice to completely determine these functions. This point of view is quite general, depending mostly on the notion of duality, but is developed here in the context of a special class of black holes for the sake of simplicity.

## 2 Automorphic black holes as probes of extra dimensions

Over the past decade impressive progress has been made toward the resolution of a problem that is almost 40 years old - the microscopic understanding of the entropy of black holes. It has proven useful to focus on black holes with extended supersymmetries because such objects are simple

enough, but not too simple. More recently, it was shown in particular for certain types of black holes in  $\mathcal{N} = 4$  supersymmetric theories that their entropy is encoded in the Fourier coefficients of Siegel modular forms, automorphic forms that provide some of the simplest generalizations of classical modular forms of one variable with respect to congruence subgroups of the full modular group  $\text{SL}(2, \mathbb{Z})$ .

The general conceptual framework of automorphic entropy functions has not been formalized yet, but a rough outline that encodes the existing concrete examples can be formulated as follows. Suppose we have a theory which contains scalar fields parametrized by a homogeneous space  $\prod_k (G_k/H_k)$  of Lie groups  $G_k$  and subgroups  $H_k$ , leading to electric and magnetic charge vectors  $Q = (Q_e, Q_m)$  in a lattice  $\Lambda$ . Assume further that the theory has a discrete T-duality group  $\prod_k D_k(\mathbb{Z})$  defined over the rational integers  $\mathbb{Z}$  and that the charge vector  $Q$  leads to norms  $\|Q\|_i$ ,  $i = 1, \dots, r$  which are T-duality invariant. Choosing variables  $\tau_i, i = 1, \dots, r$  conjugate to the invariant charge norms  $\|Q\|_i$  one can consider automorphic forms  $\Phi(\tau_i)$  defined on the resulting generalized upper half plane  $\mathcal{H}_r$  spanned by the  $\tau_i$ .

The idea is that by endowing the charge norms with an appropriate integral structure  $\mathbb{Z} \ni k_i \sim \|Q\|_i$ , the Fourier expansion of such automorphic forms

$$\Phi(\tau_i) = \sum_{k_n} g(k_1, \dots, k_r) q_1^{k_1} \cdots q_r^{k_r}, \quad q_k := e^{2\pi i \tau_k}, \quad (1)$$

determines the automorphic entropy of the black holes. Writing the expansion of the partition function  $Z(\tau_i)$  as

$$Z(\tau_i) = \frac{1}{\tilde{\Phi}(\tau_i)} = \sum_{k_n} d(k_1, \dots, k_r) q_1^{k_1} \cdots q_r^{k_r}, \quad (2)$$

leads to the microscopic entropy as a function of the charges as

$$S_{\text{mic}}(Q) \sim \ln d(\|Q\|_1, \dots, \|Q\|_r). \quad (3)$$

Here  $\tilde{\Phi}$  denotes a slight modification of the Siegel form  $\Phi$  that is determined by the vanishing behavior of  $\Phi$ .

The above outline encompasses the behavior of the entropy of black holes in certain  $\mathcal{N} = 4$  compactifications obtained by considering  $\mathbb{Z}_N$ -quotients of the heterotic toroidal compactification  $\text{Het}(T^6)$  (or their type IIA duals), a class of models first considered in ref. [3]. Specif-

ically, it was shown in [4, 5, 6] that for these  $\text{CHL}_N$  models the microscopic entropy of extreme Reissner-Nordström type black holes is described by genus two Siegel modular forms  $\Phi^N(\tau_1, \tau_2, \tau_3) \in S_w(\Gamma_0^{(2)}(N))$ , where the weight  $w$  is determined by the order  $N$  of the quotient group. Here the automorphic groups are Hecke type congruence subgroups of the symplectic group,  $\Gamma_0^{(2)}(N) \subset \text{Sp}(4, \mathbb{Z})$ .

### 3 From Siegel entropy to automorphic geometry

Given black holes whose entropy is determined by automorphic forms one can ask whether the compactification manifold leads to motives rich enough to support these automorphic forms. While it is not expected that general automorphic forms are of motivic origin, the subset of forms of algebraic type are conjectured to have a geometric origin [7, 8].

In the special case of genus two Siegel forms the conjectures concerning their motivic origin indicate that the compactification manifold cannot provide the appropriate motivic cycle structure if one follows the picture developed in arithmetic geometry. The quickest way to see this is by noting that the Hodge decomposition of a pure spinor motive  $M_\Phi$  associated to a genus two Siegel form  $\Phi$  takes the form

$$H(M_\Phi) = H^{2w-3,0} \oplus H^{w-1,w-2} \oplus H^{w-2,w-1} \oplus H^{0,2w-3}. \quad (4)$$

For mixed motives this relation can change even for classical forms [9]. While the Hodge type (4) is that of a Calabi-Yau variety, the precise structure is only correct for modular forms of weight three, which is too restrictive for the  $\text{CHL}_N$  Siegel modular forms whose weights can be as high as  $w = 10$ . Hence for most  $\text{CHL}_N$  models the Siegel modular forms cannot be induced directly by motives in the way usually envisioned in mathematics.

In the face of this first obstruction it is useful to recognize that the Siegel modular forms which appear in the context of  $\text{CHL}_N$  black hole entropy are not of general type. Instead, they are obtained by combining the Skoruppa lift [10] from classical modular forms to Jacobi forms with the Maaß lift [11] from Jacobi forms to Siegel modular forms

$$f_{w+2}(\tau) \in S_{w+2} \xrightarrow{\text{SL}} \varphi_{w,1}(\tau, \rho) \in J_w \xrightarrow{\text{ML}} \Phi_w(\tau, \sigma, \rho) \in S_w. \quad (5)$$

While the Maaß-Skoruppa lift represents an important step, it does not immediately solve the problem because the motivic support  $M_f$  for classical modular forms  $f$  of weight  $w$  leads to the Hodge decomposition

$$H(M_f) = H^{w-1,0} \oplus H^{0,w-1}, \quad (6)$$

hence the only modular forms that can fit into heterotic compactifications have weight two, three, or four. Both of the above obstructions to a geometric interpretation of black hole automorphic forms can be overcome, as described in the remainder of this paper.

The key to the identification of the motivic origin of the  $\text{CHL}_N$  black hole entropy turns out to be an additional lift construction that interprets the Maaß-Skoruppa roots  $f_{w+2}$  in terms of modular forms of weight two for arbitrary  $N$ . It can be shown that the set of Maaß-Skoruppa roots of the  $\text{CHL}_N$  models decomposes into two distinct classes of forms, one class admitting complex multiplication, the other not. For this reason it is clear that the lifts of weight two modular forms to the higher weight forms  $f_{w+2}$  must involve two different constructions, depending on the type of the higher weight forms  $f_{w+2}$ . For the forms without complex multiplication the lift interpretation of  $f_{w+2}$  in terms of weight two form  $f_2 \in S_2$  can be shown to be given by the following relation

$$f_{w+2}(q) = f_2(q^{1/m})^m, \quad \text{with } m = \frac{1}{2} \left\lceil \frac{24}{N+1} \right\rceil \in \mathbb{N}, \quad (7)$$

where  $\lceil \cdot \rceil$  is the ceiling function. The lift for the class of Maaß-Skoruppa roots with complex multiplication derives from the existence of algebraic Hecke characters  $\psi_H$  of weight one. The  $L$ -functions of powers  $\psi_H^{w+1}$  of these characters are the inverse Mellin transforms of the forms  $f_{w+2}$  [12].

The interpretation of all  $\text{CHL}_N$  Maaß-Skoruppa roots  $f_{w+2}$  in terms of weight two modular forms  $f_2^{\tilde{N}}$  via the additional lifts just described implies that the motivic origin of the Siegel modular entropy of  $\text{CHL}_N$  models is to be found in elliptic curves, as opposed to higher dimensional geometric structures associated to the compactification manifold. This follows from the fact that for each of the models it is possible to find an elliptic curve  $E_{\tilde{N}}$ , whose conductor  $\tilde{N}$  depends on the order  $N$  of the quotient group  $\mathbb{Z}_N$ , such that the modular form  $f(E_{\tilde{N}}, q)$  associated to  $E_{\tilde{N}}$  via its  $L$ -function agrees with the modular form  $f_2^{\tilde{N}}$ :

$$f_2^{\tilde{N}}(q) = f(E, q). \quad (8)$$

Abstractly, this follows from the proof of by Wiles et. al. of the Taniyama-Shimura-Weil conjecture [13, 14, 15], but no such heavy machinery is necessary for the concrete cases based on the  $\text{CHL}_N$  models, where the elliptic curves  $E_{\tilde{N}}$  can be constructed explicitly for each order  $N$  [12].

The enhanced lift construction

$$f_2^{\tilde{N}} \longrightarrow f_{w+2}^N \longrightarrow \varphi_w^N \longrightarrow \Phi_w^N, \quad (9)$$

extending the Maaß-Skoruppa lift (5), in combination with the modular identity (8) shows that the motivic origin of the Siegel black hole entropy of the class of  $\text{CHL}_N$  models arises from lower-dimensional cycles. This is surprising because one might have expected that it is the three-dimensional structure of the compactification manifold  $X_N = T^6/\mathbb{Z}_N$  in the heterotic frame, or  $(K3 \times T^2)/\mathbb{Z}_N$  in the type IIA frame, that supports the black hole entropy of the  $\text{CHL}_N$  black holes.

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